

# THE SIZE OF COEFFICIENTS OF CERTAIN POLYNOMIALS RELATED TO THE GOLDBACH CONJECTURE

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**ABSTRACT.** Recent work of Borwein, Choi, and the second author examined a collection of polynomials closely related to the Goldbach conjecture: the polynomial  $F_N$  is divisible by the  $N$ th cyclotomic polynomial if and only if there is no representation of  $N$  as the sum of two odd primes. The coefficients of these polynomials stabilize, as  $N$  grows, to a fixed sequence  $a(m)$ ; they derived upper and lower bounds for  $a(m)$ , and an asymptotic formula for the summatory function  $A(M)$  of the sequence, both under the assumption of a famous conjecture of Hardy and Littlewood. In this article we improve these results: we obtain an asymptotic formula for  $a(m)$  under the same assumption, and we establish the asymptotic formula for  $A(M)$  unconditionally.

## 1. INTRODUCTION

Let  $R(n)$  denote the number of representations of  $n$  as the sum of two odd primes. That is,  $R(n)$  is the number of ordered pairs  $(p, q)$  of odd primes satisfying  $p + q = n$ . Of course  $R(n) = 0$  when  $n$  is odd, while the celebrated Goldbach conjecture is equivalent to the statement that  $R(n) \geq 1$  for all even integers  $n \geq 6$ . Subsequently, define

$$a(m) = \sum_{d|m} R(d);$$

these quantities are closely related to a sequence of polynomials, which we describe shortly, that have a surprising connection to the Goldbach conjecture. Also define

$$A(M) = \sum_{m=1}^M a(2m) = \sum_{m=1}^{2M} a(m),$$

a summatory function that encodes the average behavior of  $a(m)$ .

The purpose of this paper is to establish two theorems concerning the sizes of  $A(M)$  and  $a(m)$  that improve results obtained by Borwein, Choi, and the second author in [1]. The first of these theorems is an asymptotic formula for  $A(M)$ .

**Theorem 1.** *For all  $M \geq 3$ ,*

$$A(M) = \frac{\pi^2 M^2}{3 \log^2 M} + O\left(\frac{M^2 \log \log M}{\log^3 M}\right).$$

We emphasize that this theorem is unconditional; by contrast, the authors of [1] established this asymptotic formula without an explicit error term, but only under the assumption of a well-known conjecture on the number of Goldbach representations of an integer  $n$ :

**Conjecture 2** (Hardy and Littlewood [2]). *As  $n$  tends to infinity,*

$$R(2n) \sim 2C_2 \frac{n}{\log^2 n} \prod_{\substack{p|n \\ p>2}} \frac{p-1}{p-2}, \quad (1)$$

where  $C_2$  is the twin primes constant

$$C_2 = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right).$$

The authors of [1] do obtain an unconditional lower bound on  $A(M)$ , namely

$$A(M) \geq M \log M + O(M). \quad (2)$$

This lower bound required the use of a deep result of Montgomery and Vaughan [3] on the exceptional set in the Goldbach conjecture, while our proof of Theorem 1 is elementary, with the deepest ingredient being the prime number theorem. The surprising gap between the asymptotic formula in Theorem 1 and the lower bound (2) can be explained by the fact that the authors of [1] actually prove the much stronger result

$$\sum_{m=1}^{2M} \#\{d \mid m : R(d) \geq 1\} \geq M \log M + O(M),$$

which does indeed imply (2), since the summand on the left-hand side is at most  $\sum_{d|m} R(d) = a(m)$ .

Our second theorem, an asymptotic formula for  $a(m)$  conditional on the aforementioned conjecture of Hardy and Littlewood, is best stated after defining the following multiplicative function.

**Definition 3.**  $J(m)$  is the multiplicative function given by the following formula: if  $2^k \parallel m$ , then

$$J(m) = \left(2 - \frac{1}{2^k}\right) \prod_{\substack{p^\ell \parallel m \\ p>2}} \left(1 - \frac{2}{p^{\ell+1}}\right) \left(1 - \frac{2}{p}\right)^{-1}.$$

Here,  $p^\ell \parallel m$  means that  $p^\ell \mid m$  but  $p^{\ell+1} \nmid m$ .

**Theorem 4.** *If Conjecture 2 is true, then*

$$a(2m) \sim \frac{2C_2 J(m)m}{\log^2 m}$$

as  $m$  tends to infinity.

The authors of [1] were able to derive from Conjecture 2 the upper and lower bounds

$$\frac{2C_2 m}{\log^2 m} \lesssim a(2m) \lesssim \frac{4C_2 m}{\log^2 m} \prod_{\substack{p^\ell \parallel m \\ p>2}} \left(1 - \frac{2}{p^{\ell+1}}\right) \left(1 - \frac{2}{p}\right)^{-1}; \quad (3)$$

we are able here to close the small gap between these bounds.

The function  $a(m)$  and the Goldbach conjecture are linked via the sequence of polynomials

$$F_N(z) = \sum_{k=0}^{N-1} \left( \sum_{\substack{n=1 \\ 2}}^{N-1} \chi_{\mathcal{P}}(n) z^{kn} \right)^2,$$

where

$$\chi_{\mathcal{P}}(n) = \begin{cases} 1, & \text{if } n \text{ is an odd prime,} \\ 0, & \text{otherwise.} \end{cases}$$

For example,

$$\begin{aligned} F_{10}(z) = & 9 + (z^7 + z^5 + z^3)^2 + (z^{14} + z^{10} + z^6)^2 + (z^{21} + z^{15} + z^9)^2 \\ & + (z^{28} + z^{20} + z^{12})^2 + (z^{35} + z^{25} + z^{15})^2 + (z^{42} + z^{30} + z^{18})^2 \\ & + (z^{49} + z^{35} + z^{21})^2 + (z^{56} + z^{40} + z^{24})^2 + (z^{63} + z^{45} + z^{27})^2. \end{aligned}$$

It is not hard to see that for  $m \geq 1$ , the coefficient of  $z^m$  in  $F_N(z)$  is a nonnegative integer that is at most  $a(m)$ , and in fact it equals  $a(m)$  for all  $N \geq m$ . For example, when expanded out

$$F_{10}(z) = 9 + z^6 + 2z^8 + 3z^{10} + \dots + z^{126},$$

reflecting the first ten values

$$(a(1), \dots, a(10)) = (0, 0, 0, 0, 0, 1, 0, 2, 0, 3).$$

In other words, the sequence of polynomials  $F_N(z) - F_N(0)$  converges coefficient-wise to the fixed formal power series  $\sum_{m=1}^{\infty} a(m)z^m$ .

Letting  $\Phi_k(z)$  denote the  $k$ th cyclotomic polynomial as usual, the authors of [1] show that  $F_{2N}(z)$  is divisible by  $\Phi_{4N}(z)$  for every positive integer  $N$ . Experimental evidence suggests:

**Conjecture 5** (Borwein, Choi, and Samuels). *For every integer  $N \geq 3$ , the polynomial  $F_{2N}(z)/\Phi_{4N}(z)$  is irreducible in  $\mathbb{Z}[z]$ .*

The relationship between  $F_N$  and the Goldbach conjecture is more than superficial, however, as the following startling theorem displays:

**Theorem 6** (Borwein, Choi, and Samuels).  *$\Phi_N(z)$  divides  $F_N(z)$  if and only if there is no representation of  $N$  as the sum of two odd primes. In particular, Conjecture 5 implies the Goldbach conjecture.*

## 2. PROOFS OF OUR RESULTS

We begin by proving Theorem 1, although first we need to devote some time to a technical lemma that counts the number of pairs of primes whose sum lies below a given bound. Afterwards, we derive Theorem 4 from Proposition 8 below.

In order to establish Theorem 1, we must first study the function

$$Q(x) = \sum_{p+q \leq x} 1,$$

where  $p$  and  $q$  always denote primes in this paper.

**Lemma 7.** *Uniformly for  $x \geq 3$ ,*

$$Q(x) = \frac{x^2}{2 \log^2 x} + O\left(\frac{x^2 \log \log x}{\log^3 x}\right).$$

*Proof.* We begin by writing

$$Q(x) = \sum_{p \leq x} \pi(x-p) = \sum_{x/\log x \leq p \leq x-\sqrt{x}} \pi(x-p) + O\left(\sum_{p \leq x/\log x} \pi(x-p) + \sum_{x-\sqrt{x} \leq p \leq x} \pi(x-p)\right).$$

Trivially  $\pi(x-p) \leq \pi(x) \leq x$ , so

$$\begin{aligned} Q(x) &= \sum_{x/\log x \leq p \leq x-\sqrt{x}} \pi(x-p) + O\left(\sum_{p \leq x/\log x} \pi(x) + \sum_{x-\sqrt{x} \leq p \leq x} x\right) \\ &= \sum_{x/\log x \leq p \leq x-\sqrt{x}} \pi(x-p) + O\left(\pi(x)\pi\left(\frac{x}{\log x}\right) + x\sqrt{x}\right) \\ &= \sum_{x/\log x \leq p \leq x-\sqrt{x}} \pi(x-p) + O\left(\frac{x^2}{\log^3 x}\right). \end{aligned} \tag{4}$$

In the main term, the prime number theorem gives

$$\sum_{x/\log x \leq p \leq x-\sqrt{x}} \pi(x-p) = \sum_{x/\log x \leq p \leq x-\sqrt{x}} \left( \text{li}(x-p) + O\left(\frac{x-p}{\log^2(x-p)}\right) \right)$$

(we could insert a better error term, but it would not improve the final result). Since  $x-p \geq \sqrt{x}$ , we have  $\log(x-p) \gg \log x$  and so

$$\begin{aligned} &= \sum_{x/\log x \leq p \leq x-\sqrt{x}} \text{li}(x-p) + O\left(\sum_{x/\log x \leq p \leq x-\sqrt{x}} \frac{x}{\log^2 x}\right) \\ &= \sum_{x/\log x \leq p \leq x-\sqrt{x}} \text{li}(x-p) + O\left(\frac{x}{\log^2 x} \pi(x)\right) \\ &= \sum_{x/\log x \leq p \leq x-\sqrt{x}} \text{li}(x-p) + O\left(\frac{x^2}{\log^3 x}\right), \end{aligned}$$

which transforms equation (4) into

$$Q(x) = \sum_{x/\log x \leq p \leq x-\sqrt{x}} \text{li}(x-p) + O\left(\frac{x^2}{\log^3 x}\right). \tag{5}$$

Using partial summation, we have

$$\begin{aligned} \sum_{x/\log x \leq p \leq x-\sqrt{x}} \text{li}(x-p) &= \int_{x/\log x}^{x-\sqrt{x}} \text{li}(x-t) d\pi(t) \\ &= \pi(x-\sqrt{x}) \text{li}(\sqrt{x}) - \pi\left(\frac{x}{\log x}\right) \text{li}\left(x - \frac{x}{\log x}\right) + \int_{x/\log x}^{x-\sqrt{x}} \frac{\pi(t)}{\log(x-t)} dt, \end{aligned}$$

since the  $t$ -derivative of  $\text{li}(x - t)$  is  $-1/\log(x - t)$ . In other words,

$$\begin{aligned} \sum_{x/\log x \leq p \leq x-\sqrt{x}} \text{li}(x - p) &= O\left(x\sqrt{x} + \pi\left(\frac{x}{\log x}\right) \text{li}(x)\right) + \int_{x/\log x}^{x-\sqrt{x}} \frac{\pi(t)}{\log(x - t)} dt \\ &= \int_{x/\log x}^{x-\sqrt{x}} \frac{\pi(t)}{\log(x - t)} dt + O\left(\frac{x^2}{\log^3 x}\right), \end{aligned}$$

and so equation (5) becomes

$$Q(x) = \int_{x/\log x}^{x-\sqrt{x}} \frac{\pi(t)}{\log(x - t)} dt + O\left(\frac{x^2}{\log^3 x}\right).$$

Using the prime number theorem again, this becomes

$$\begin{aligned} Q(x) &= \int_{x/\log x}^{x-\sqrt{x}} \frac{1}{\log(x - t)} \left( \frac{t}{\log t} + O\left(\frac{t}{\log^2 t}\right) \right) dt + O\left(\frac{x^2}{\log^3 x}\right) \\ &= \int_{x/\log x}^{x-\sqrt{x}} \frac{t}{(\log t) \log(x - t)} dt + O\left( \int_{x/\log x}^{x-\sqrt{x}} \frac{t}{(\log^2 t) \log(x - t)} dt + \frac{x^2}{\log^3 x} \right). \end{aligned} \quad (6)$$

In the error term, again  $\log(x - t) \gg \log x$  and  $\log^2 t \gg \log^2 x$  due to the endpoints of integration, and so the entire integral is  $\ll x^2/\log^3 x$ . In the main term, we have

$$\log x \geq \log t \geq \log \frac{x}{\log x} = \log x - \log \log x = (\log x) \left( 1 + O\left(\frac{\log \log x}{\log x}\right) \right),$$

and therefore equation (6) becomes

$$Q(x) = \frac{1}{\log x} \left( 1 + O\left(\frac{\log \log x}{\log x}\right) \right) \int_{x/\log x}^{x-\sqrt{x}} \frac{t}{\log(x - t)} dt + O\left(\frac{x^2}{\log^3 x}\right). \quad (7)$$

Finally,

$$\begin{aligned} \int_{x/\log x}^{x-\sqrt{x}} \frac{t}{\log(x - t)} dt &= \int_0^{x-2} \frac{t}{\log(x - t)} dt + O\left( \int_0^{x/\log x} t dt + \int_{x-\sqrt{x}}^{x-2} t dt \right) \\ &= \int_2^x \frac{x - u}{\log u} du + O\left(\frac{x^2}{\log^2 x}\right) \\ &= x \text{li}(x) - \int_2^x \frac{u}{\log u} du + O\left(\frac{x^2}{\log^2 x}\right). \end{aligned} \quad (8)$$

By integration by parts, this integral is

$$\begin{aligned} \int_2^x \frac{u}{\log u} du &= \frac{u^2}{2} \frac{1}{\log u} \Big|_2^x + \int_2^x \frac{u^2}{2} \frac{1}{u \log^2 u} du \\ &= \frac{x^2}{2 \log x} + O\left( 1 + \int_2^{\sqrt{x}} \frac{u}{\log^2 u} du + \int_{\sqrt{x}}^x \frac{u}{\log^2 u} du \right) \\ &= \frac{x^2}{2 \log x} + O\left( \sqrt{x} \cdot x + x \frac{x}{\log^2 x} \right) = \frac{x^2}{2 \log x} + O\left(\frac{x^2}{\log^2 x}\right). \end{aligned}$$

Therefore equation (8) becomes

$$\int_{x/\log x}^{x-\sqrt{x}} \frac{t}{\log(x-t)} dt = x \operatorname{li}(x) - \frac{x^2}{2 \log x} + O\left(\frac{x^2}{\log^2 x}\right) = \frac{x^2}{2 \log x} + O\left(\frac{x^2}{\log^2 x}\right)$$

by the fact that  $\operatorname{li}(x) = x/\log x + O(x/\log^2 x)$ . Using this in equation (7) finally yields

$$\begin{aligned} Q(x) &= \frac{1}{\log x} \left(1 + O\left(\frac{\log \log x}{\log x}\right)\right) \left(\frac{x^2}{2 \log x} + O\left(\frac{x^2}{\log^2 x}\right)\right) + O\left(\frac{x^2}{\log^3 x}\right) \\ &= \frac{x^2}{2 \log^2 x} + O\left(\frac{x^2 \log \log x}{\log^3 x}\right), \end{aligned}$$

as claimed.  $\square$

Equipped with Lemma 7, we are now prepared to prove Theorem 1.

*Proof of Theorem 1.* Starting with the definitions of  $a(m)$  and  $A(M)$ , we have

$$A(M) = \sum_{m=1}^{2M} a(m) = \sum_{m=1}^{2M} \sum_{d|m} R(d) = \sum_{m=1}^{2M} \sum_{d|m} \sum_{p+q=d} 1 = \sum_{p+q \leq 2M} \sum_{\substack{1 \leq m \leq 2M \\ (p+q)|m}} 1.$$

Writing  $m = (p+q)n$ , we obtain

$$A(M) = \sum_{p+q \leq 2M} \sum_{1 \leq n \leq 2M/(p+q)} 1 = \sum_{1 \leq n \leq M/2} \sum_{p+q \leq 2M/n} 1 = \sum_{1 \leq n \leq M/2} Q\left(\frac{2M}{p+q}\right). \quad (9)$$

The trivial bound  $Q(x) \leq x^2$  allows us to write

$$A(M) = \sum_{1 \leq n \leq \log^3 M} Q\left(\frac{2M}{n}\right) + O\left(\sum_{n > \log^3 M} \left(\frac{2M}{n}\right)^2\right) = \sum_{1 \leq n \leq \log^3 M} Q\left(\frac{2M}{n}\right) + O\left(\frac{M^2}{\log^3 M}\right),$$

since  $\sum_{n > \log^3 M} n^{-2} \ll 1/\log^3 M$  by comparison with an integral. We use Lemma 7 to get

$$\begin{aligned} A(M) &= \sum_{1 \leq n \leq \log^3 M} \left( \frac{(2M/n)^2}{2 \log^2(2M/n)} + O\left(\frac{(2M/n)^2 \log \log(2M/n)}{\log^3(2M/n)}\right) \right) + O\left(\frac{M^2}{\log^3 M}\right) \\ &= 2M^2 \sum_{1 \leq n \leq \log^3 M} \frac{1}{\log^2(2M/n)} \frac{1}{n^2} + O\left(\sum_{1 \leq n \leq \log^3 M} \frac{\sqrt{2M} \log \log 2M}{\log^3 2M} \left(\frac{2M}{n}\right)^{3/2} + \frac{M^2}{\log^3 M}\right), \end{aligned}$$

since  $\sqrt{x} \log \log x / \log^3 x$  is an (eventually) increasing function of  $x$ . By the convergence of  $\sum_n n^{-3/2}$ , we obtain

$$A(M) = 2M^2 \sum_{1 \leq n \leq \log^3 M} \frac{1}{\log^2(2M/n)} \frac{1}{n^2} + O\left(\frac{M^2 \log \log M}{\log^3 M}\right).$$

Finally, we have  $\log(2M/n) = \log M - \log(n/2) = \log M + O(\log(\log^3 M)) = (\log M)(1 + O(\log \log M / \log M))$  as before. Therefore

$$A(M) = \frac{2M^2}{\log^2 M} \left(1 + O\left(\frac{\log \log M}{\log M}\right)\right) \sum_{1 \leq n \leq \log^3 M} \frac{1}{n^2} + O\left(\frac{M^2 \log \log M}{\log^3 M}\right).$$

We conclude that

$$\begin{aligned} A(M) &= \frac{2M^2}{\log^2 M} \left( 1 + O\left(\frac{\log \log M}{\log M}\right) \right) \left( \zeta(2) + O\left(\frac{1}{\log^3 M}\right) \right) + O\left(\frac{M^2 \log \log M}{\log^3 M}\right) \\ &= \frac{\pi^2 M^2}{3 \log^2 M} + O\left(\frac{M^2 \log \log M}{\log^3 M}\right), \end{aligned}$$

as desired.  $\square$

We now move on to a proposition from which we will deduce Theorem 4. Define

$$f(n) = \prod_{\substack{p|n \\ p>2}} \frac{p-1}{p-2}$$

to be the multiplicative function appearing in Conjecture 2, and note that if  $k \geq 0$  is the integer such that  $2^k \parallel m$ , then

$$\begin{aligned} \sum_{d|m} df(d) &= \prod_{p^\ell \parallel m} \sum_{d|p^\ell} df(d) \\ &= \prod_{p^\ell \parallel m} (1 + pf(p) + p^2 f(p^2) + \cdots + p^\ell f(p^\ell)) \\ &= \left( 1 + 2 \frac{2^k - 1}{2 - 1} \right) \prod_{\substack{p^\ell \parallel m \\ p>2}} \left( 1 + \frac{p-1}{p-2} \cdot p \frac{p^\ell - 1}{p-1} \right) \\ &= (2^{k+1} - 1) \prod_{\substack{p^\ell \parallel m \\ p>2}} \frac{p^{\ell+1} - 2}{p-2} = mJ(m) \end{aligned} \tag{10}$$

by comparison with Definition 3.

**Proposition 8.** *Let  $0 < \varepsilon \leq \frac{1}{2}$  be given. Suppose there exists a positive integer  $n(\varepsilon)$  such that*

$$(1 - \varepsilon)2C_2 f(n) \frac{n}{\log^2 n} \leq R(2n) \leq (1 + \varepsilon)2C_2 f(n) \frac{n}{\log^2 n} \tag{11}$$

*for all  $n > n(\varepsilon)$ . Then there exists a constant  $m(\varepsilon)$  such that*

$$(1 - 2\varepsilon)2C_2 J(m) \frac{m}{\log^2 m} \leq a(2m) \leq (1 + 11\varepsilon)2C_2 J(m) \frac{m}{\log^2 m} \tag{12}$$

*for all  $m > m(\varepsilon)$ .*

It is clear that Theorem 4 follows from Proposition 8, since Conjecture 2 implies that the hypothesis of Proposition 8 holds for every  $\varepsilon > 0$ .

*Proof of Proposition 8.* We shall not keep track explicitly of the necessary value for  $m(\varepsilon)$ , instead simply saying “when  $m$  is large enough” (in terms of  $\varepsilon$ ) in the appropriate places. We begin by writing

$$a(2m) = \sum_{c|2m} R(c) = \sum_{d|m} R(2d) = \sum_{\substack{d|m \\ d \leq m^{1-\varepsilon}}} R(2d) + \sum_{\substack{d|m \\ d > m^{1-\varepsilon}}} R(2d) \tag{13}$$

(where the second equality uses the fact that  $R(c) = 0$  when  $c$  is odd).

First we establish the upper bound in equation (12). We have  $m^{1-\varepsilon} > n(\varepsilon)$  when  $m$  is large enough, and so the summands in the second sum on the right-hand side of equation (13) can be bounded above by the upper bound in equation (11). For the first sum on the right-hand side we simply use the trivial bound  $R(2n) \leq n$ . The result is

$$\begin{aligned} a(2m) &\leq \sum_{\substack{d|m \\ d \leq m^{1-\varepsilon}}} d + \sum_{\substack{d|m \\ d > m^{1-\varepsilon}}} (1+\varepsilon)2C_2 f(d) \frac{d}{\log^2 d} \\ &\leq \sum_{\substack{d|m \\ d \leq m^{1-\varepsilon}}} m^{1-\varepsilon} + (1+\varepsilon)2C_2 \frac{1}{(1-\varepsilon)^2 \log^2 m} \sum_{\substack{d|m \\ d > m^{1-\varepsilon}}} df(d) \\ &= m^{1-\varepsilon} \tau(m) + \frac{1+\varepsilon}{(1-\varepsilon)^2} \frac{2C_2}{\log^2 m} mJ(m) \end{aligned}$$

using the identity (10), where  $\tau(m)$  denotes the number of divisors of  $m$ . It is well known that  $\tau(m) \ll_\varepsilon m^{\varepsilon/3}$ , and so the first term is less than  $\varepsilon m / \log^2 m$  when  $m$  is large enough. Also  $(1+\varepsilon)/(1-\varepsilon)^2 \leq 1+10\varepsilon$  for  $0 < \varepsilon \leq \frac{1}{2}$ . Therefore

$$a(2m) \leq \varepsilon \frac{m}{\log^2 m} + (1+10\varepsilon) \frac{2C_2}{\log^2 m} mJ(m) \leq (1+11\varepsilon) 2C_2 J(m) \frac{m}{\log^2 m}$$

when  $m$  is large enough, since  $J(m) \geq 1$  for all positive integers  $m$  and  $2C_2 > 1$ . This establishes the upper bound in equation (12).

A similar method addresses the lower bound in equation (12). Since  $m^{1-\varepsilon} > n(\varepsilon)$  when  $m$  is large enough, the summands in the second sum on the right-hand side of equation (13) can be bounded below by the lower bound in equation (11); the first sum on the right-hand side is nonnegative, and so we can simply delete it. We obtain the lower bound

$$\begin{aligned} a(2m) &\geq \sum_{\substack{d|m \\ d > m^{1-\varepsilon}}} (1+\varepsilon)2C_2 f(d) \frac{d}{\log^2 d} \\ &\geq (1-\varepsilon) \frac{2C_2}{\log^2 m} \sum_{\substack{d|m \\ d > m^{1-\varepsilon}}} df(d) = (1-\varepsilon) \frac{2C_2}{\log^2 m} \left( mJ(m) - \sum_{\substack{d|m \\ d \leq m^{1-\varepsilon}}} df(d) \right), \end{aligned} \quad (14)$$

again using the identity (10). This last sum is bounded above by

$$\sum_{\substack{d|m \\ d \leq m^{1-\varepsilon}}} df(d) \leq \sum_{d|m} \left( \frac{m^{1-\varepsilon}}{d} \right)^{1+\varepsilon/2} df(d) \leq m^{1-\varepsilon/2} \sum_{d|m} \prod_{\substack{p|d \\ p>2}} \frac{p-1}{p^{\varepsilon/2}(p-2)}.$$

There are only finitely many primes  $p$  for which  $(p-1)/p^{\varepsilon/2}(p-2)$  exceeds 1, and so the inner product on the right-hand side is uniformly bounded by some constant  $C(\varepsilon)$ . Therefore

$$\sum_{\substack{d|m \\ d \leq m^{1-\varepsilon}}} df(d) \leq C(\varepsilon) m^{1-\varepsilon/2} \sum_{d|m} 1 = C(\varepsilon) m^{1-\varepsilon/2} \tau(m),$$



which as above is less than  $\varepsilon m$  for  $m$  large enough. Therefore equation (14) becomes

$$a(m) \geq (1 - \varepsilon) \frac{2C_2}{\log^2 m} (mJ(m) - \varepsilon m) \geq (1 - 2\varepsilon) 2C_2 J(m) \frac{m}{\log^2 m}$$

when  $m$  is large enough, again since  $J(m) \geq 1$  always. This establishes the lower bound in equation (12).  $\square$

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